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Classe : 7 C$_1$

Corrigé du Bac 2015 SC

Exercice : 4

Soit : $f(x) = \frac{x+1}{e^x} = (x+1)e^{-x}$

1) a) $D_f =]-\infty, +\infty[$

- $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} (x+1)e^{-x} = -\infty \cdot +\infty = -\infty$

$\Rightarrow \boxed{\lim_{x\to-\infty} f(x) = -\infty}$

- $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x+1}{e^x} = \lim_{+\infty} \frac{x}{e^x} + \frac{1}{e^x}$

$= 0 + 0 = 0 \quad \Rightarrow \boxed{\lim_{x\to+\infty} f(x) = 0}$

$f'(x) = e^{-x} - (x+1)e^{-x}$

$f'(x) = e^{-x} - xe^{-x} - e^{-x} = -xe^{-x}$

$\Rightarrow \boxed{f'(x) = -xe^{-x}}$

$f'(x) = 0 \Rightarrow -x = 0 \Rightarrow \boxed{x = 0}$

$f(0) = 1$

T.V

x	$-\infty$	0	$+\infty$
$f'(x)$	+	$\vert$	+
$f(x)$	$-\infty$ ↗	1	↘ 0

b) $\mathcal{C} \cap ox$: $f(x) = 0 \Rightarrow x+1 = 0 \Rightarrow \boxed{x = -1}$

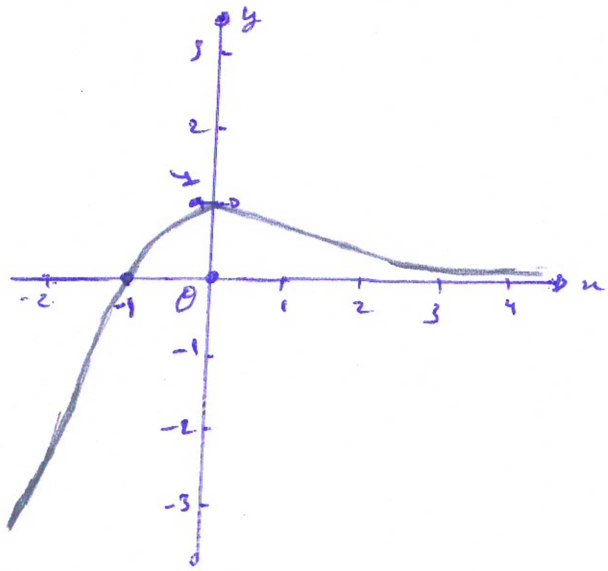
$A(-1, 0)$

$\mathcal{C} \cap oy$: $f(0) = 1 \Rightarrow B(0, 1)$

Branche infinies :

- $y = 0$ AH au vasinage de $+\infty$
- $\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \left[\frac{x+1}{x} \cdot e^{-x}\right] = +\infty$

$\mathcal{C}$ admet une branche parabolique de direction (oy) !



2) $f_n(x) = \frac{(1+x)^n}{e^x} = (1+x)^n e^{-x}$

$\forall x \in \mathbb{R}$ $F_n(x) = \int_{-1}^{x} f_n(t)\,dt$

$F_{n+1}(x) = \int_{-1}^{x} (1+x)^{n+1} e^{-x}$

On pose : $\begin{cases} u(x) = (1+t)^{n+1} \to u'(x) = (n+1)(1+t)^n \\ v'(x) = e^{-t} \to v(x) = -e^{-t} \end{cases}$

$F_{n+1}(x) = \left[-(1+t)^{n+1} \cdot e^{-t}\right]_{-1}^{x} + (n+1)\int_{-1}^{x} (1+t)^{n} e^{-t}\,dt$

$\boxed{F_{n+1}(x) = (n+1)\,F_n(x) - f_{n+1}(x)}$

Exercice 4 (suite)

3) $I_n = F_n(0) = \int_{-1}^{0} f_n(t)\,dt$

a) on a : $F_{n+1}(x) = (n+1)F_n(x) - f_{n+1}(x)$

On prend $x = 0$

$F_{n+1}(0) = (n+1)F_n(0) - f_{n+1}(0)$

$$\boxed{I_{n+1} = (n+1)I_n - 1}$$

b) on rappelle que :

$\forall x \in [0,1]$ $\quad 0 \leq x^{n+1} \leq x^n \leq 1$

$-1 \leq t \leq 0 \Rightarrow 0 \leq 1+t \leq 1$

$\Rightarrow 0 \leq (1+t)^{n+1} \leq (1+t)^n$

$0 \leq (1+t)^{n+1} e^{-t} \leq (1+t)^n e^{-t}$

$0 \leq \int_{-1}^{0} (1+t)^{n+1} e^{-t} \leq \int_{-1}^{0} (1+t)^n e^{-t}$

$$\boxed{0 \leq I_{n+1} \leq I_n}$$

d'où : (I_n) et décroissante et positive

d'une par : $I_{n+1} \leq I_n$ car $I_n \searrow$

$(n+1)I_n - 1 \leq I_n \Rightarrow nI_n + I_n - I_n \leq 1$

$\Rightarrow nI_n \leq 1 \Rightarrow \boxed{I_n \leq \frac{1}{n}}$

d'autre par I_n positive $\Rightarrow I_{n+1} \geq 0$

$\Rightarrow 0 \leq (n+1)I_n - 1 \Rightarrow \boxed{\frac{1}{n+1} \leq I_n}$

donc : $\boxed{\frac{1}{n+1} \leq I_n \leq \frac{1}{n}}$

On en déduit que $\lim\limits_{n \to +\infty} I_n = 0$

4) $\forall n \geq 1$ $\quad U_n = \frac{I_n}{n!}$

on a : $I_{n+1} = (n+1)I_n - 1$

$\Rightarrow \frac{I_{n+1}}{(n+1)!} = \frac{(n+1)I_n}{(n+1)!} - \frac{1}{(n+1)!}$

$\Rightarrow \boxed{U_{n+1} = U_n - \frac{1}{(n+1)!}}$

On a : $U_1 = \frac{I_1}{1!} = I_1 = \int_{-1}^{0} (1+t)e^{-t}\,dt$

$\begin{cases} u = 1+t \longrightarrow u' = 1 \\ v' = e^{-t} \longrightarrow v = -e^{-t} \end{cases}$

$U_1 = \left[-(1+t)e^{-t}\right]_{-1}^{0} + \int_{-1}^{0} e^{-t}\,dt$

$U_1 = \left[-(1+t)e^{-t}\right]_{-1}^{0} + \left[-e^{-t}\right]_{-1}^{0}$

$U_1 = -1 - 1 + e \Rightarrow \boxed{U_1 = e - 2}$

on a : $U_{n+1} = U_n - \frac{1}{(n+1)!}$ Alors

$U_2 = U_1 - \frac{1}{2!}$

$U_3 = U_2 - \frac{1}{3!}$

$U_4 = U_3 - \frac{1}{4!}$

$\vdots$

$U_n = U_{n-1} - \frac{1}{n!}$

par addition : $U_n = U_1 - \left(\frac{1}{2!} + \frac{1}{3!} \cdots \frac{1}{n!}\right)$

$U_n = e - 2 - \left(\frac{1}{2!} + \frac{1}{3!} \cdots \frac{1}{n!}\right)$

$U_n = e - \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdots \frac{1}{n!}\right)$

$$\boxed{U_n = e - \sum_{k=0}^{n} \frac{1}{k!}}$$

b) Calcul $\lim\limits_{n \to +\infty} \sum_{k=0}^{n} \frac{1}{k!}$

On sait que $\lim\limits_{n \to +\infty} U_n = \lim\limits_{+\infty} \frac{I_n}{n} = 0$

$\sum_{k=0}^{n} \frac{1}{k!} = e - U_n$

d'où : $\boxed{\lim\limits_{n \to +\infty} \sum_{k=0}^{n} \frac{1}{k!} = e}$